

On three-parametric Lie groups as quasi-Kähler manifolds with Killing Norden metric

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A 3-parametric family of 6-dimensional quasi-Kähler manifolds with Norden metric is constructed on a Lie group. This family is characterized geometrically. The condition for such a 6-manifold to be isotropic Kähler is given.*

Keywords: almost complex manifold, Norden metric, quasi-Kähler manifold, indefinite metric, non-integrable almost complex structure, Lie group

Introduction

It is a fundamental fact that on an almost complex manifold with Hermitian metric (almost Hermitian manifold), the action of the almost complex structure on the tangent space at each point of the manifold is isometry. There is another kind of metric, called a Norden metric or a B -metric on an almost complex manifold, such that the action of the almost complex structure is anti-isometry with respect to the metric. Such a manifold is called an almost complex manifold with Norden metric¹ or with B -metric.² See also Ref. 5 for generalized B -manifolds. It is known¹ that these manifolds are classified into eight classes.

The purpose of the present paper is to exhibit, by construction, almost complex structures with Norden metric on Lie groups as 6-manifolds, which are of a certain class, called quasi-Kähler manifold with Norden metric. It is proved that the constructed 6-manifold is isotropic Kählerian³ if and only if it is scalar flat or it has zero holomorphic sectional curvatures.

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1. Almost Complex Manifolds with Norden Metric

1.1. Preliminaries

Let (M, J, g) be a $2n$ -dimensional almost complex manifold with Norden metric, i. e. J is an almost complex structure and g is a metric on M such that

$$J^2 X = -X, \quad g(JX, JY) = -g(X, Y) \quad (1)$$

for all differentiable vector fields X, Y on M , i. e. $X, Y \in \mathfrak{X}(M)$.

The associated metric \tilde{g} of g on M given by $\tilde{g}(X, Y) = g(X, JY)$ for all $X, Y \in \mathfrak{X}(M)$ is a Norden metric, too. Both metrics are necessarily of signature (n, n) . The manifold (M, J, \tilde{g}) is an almost complex manifold with Norden metric, too.

Further, X, Y, Z, U (x, y, z, u , respectively) will stand for arbitrary differentiable vector fields on M (vectors in $T_p M$, $p \in M$, respectively).

The Levi-Civita connection of g is denoted by ∇ . The tensor field F of type $(0, 3)$ on M is defined by

$$F(X, Y, Z) = g((\nabla_X J)Y, Z). \quad (2)$$

It has the following symmetries

$$F(X, Y, Z) = F(X, Z, Y) = F(X, JY, JZ). \quad (3)$$

Further, let $\{e_i\}$ ($i = 1, 2, \dots, 2n$) be an arbitrary basis of $T_p M$ at a point p of M . The components of the inverse matrix of g are denoted by g^{ij} with respect to the basis $\{e_i\}$.

The Lie form θ associated with F is defined by

$$\theta(z) = g^{ij} F(e_i, e_j, z). \quad (4)$$

A classification of the considered manifolds with respect to F is given in Ref. 1. Eight classes of almost complex manifolds with Norden metric are characterized there according to the properties of F . The three basic classes are given as follows

$$\begin{aligned} \mathcal{W}_1 : F(x, y, z) &= \frac{1}{4n} \{g(x, y)\theta(z) + g(x, z)\theta(y) \\ &\quad + g(x, Jy)\theta(Jz) + g(x, Jz)\theta(Jy)\}; \\ \mathcal{W}_2 : \mathfrak{S}_{x, y, z} F(x, y, Jz) &= 0, \quad \theta = 0; \\ \mathcal{W}_3 : \mathfrak{S}_{x, y, z} F(x, y, z) &= 0, \end{aligned} \quad (5)$$

where \mathfrak{S} is the cyclic sum by three arguments.

The special class \mathcal{W}_0 of the Kähler manifolds with Norden metric belonging to any other class is determined by the condition $F = 0$.

1.2. Curvature properties

Let R be the curvature tensor field of ∇ defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z. \quad (6)$$

The corresponding tensor field of type $(0, 4)$ is determined as follows

$$R(X, Y, Z, U) = g(R(X, Y)Z, U). \quad (7)$$

The Ricci tensor ρ and the scalar curvature τ are defined as usual by

$$\rho(y, z) = g^{ij} R(e_i, y, z, e_j), \quad \tau = g^{ij} \rho(e_i, e_j). \quad (8)$$

Let $\alpha = \{x, y\}$ be a non-degenerate 2-plane (i. e. $\pi_1(x, y, y, x) = g(x, x)g(y, y) - g(x, y)^2 \neq 0$) spanned by vectors $x, y \in T_p M, p \in M$. Then, it is known, the sectional curvature of α is defined by the following equation

$$k(\alpha) = k(x, y) = \frac{R(x, y, y, x)}{\pi_1(x, y, y, x)}. \quad (9)$$

The basic sectional curvatures in $T_p M$ with an almost complex structure and a Norden metric g are

- *holomorphic sectional curvatures* if $J\alpha = \alpha$;
- *totally real sectional curvatures* if $J\alpha \perp \alpha$ with respect to g .

1.3. Isotropic Kähler manifolds

The square norm $\|\nabla J\|$ of ∇J is defined in Ref. 3 by

$$\|\nabla J\| = g^{ij} g^{kl} g((\nabla_{e_i} J) e_k, (\nabla_{e_j} J) e_l). \quad (10)$$

Having in mind the definition (2) of the tensor F and the properties (3), we obtain the following equation for the square norm of ∇J

$$\|\nabla J\| = g^{ij} g^{kl} g^{pq} F_{ikp} F_{jlq}, \quad (11)$$

where $F_{ikp} = F(e_i, e_k, e_p)$.

Definition 1.1 ⁽⁶⁾. *An almost complex manifold with Norden metric satisfying the condition $\|\nabla J\| = 0$ is called an isotropic Kähler manifold with Norden metric.*

Remark 1.1. It is clear, if a manifold belongs to the class \mathcal{W}_0 , then it is isotropic Kählerian but the inverse statement is not always true.

2. Lie groups as Quasi-Kähler manifolds with Killing Norden metric

The only class of the three basic classes, where the almost complex structure is not integrable, is the class \mathcal{W}_3 – the class of the *quasi-Kähler manifolds with Norden metric*.

Let us remark that the definitional condition from (5) implies the vanishing of the Lie form θ for the class \mathcal{W}_3 .

Let V be a $2n$ -dimensional vector space and consider the structure of the Lie algebra defined by the brackets $[E_i, E_j] = C_{ij}^k E_k$, where $\{E_1, E_2, \dots, E_{2n}\}$ is a basis of V and $C_{ij}^k \in \mathbb{R}$.

Let G be the associated connected Lie group and $\{X_1, X_2, \dots, X_{2n}\}$ be a global basis of left invariant vector fields induced by the basis of V . Then the Jacobi identity has the form

$$\mathfrak{S}_{X_i, X_j, X_k} [[X_i, X_j], X_k] = 0. \quad (12)$$

Next we define an almost complex structure by the conditions

$$JX_i = X_{n+i}, \quad JX_{n+i} = -X_i, \quad i \in \{1, 2, \dots, n\}. \quad (13)$$

Let us consider the left invariant metric defined by the following way

$$\begin{aligned} g(X_i, X_i) &= -g(X_{n+i}, X_{n+i}) = 1, & i \in \{1, 2, \dots, n\} \\ g(X_j, X_k) &= 0, & j \neq k \in \{1, 2, \dots, 2n\}. \end{aligned} \quad (14)$$

The introduced metric is a Norden metric because of (13).

In this way, the induced $2n$ -dimensional manifold (G, J, g) is an almost complex manifold with Norden metric, in short *almost Norden manifold*.

The condition the Norden metric g be a Killing metric of the Lie group G with the corresponding Lie algebra \mathfrak{g} is $g(\text{ad}X(Y), Z) = -g(Y, \text{ad}X(Z))$, where $X, Y, Z \in \mathfrak{g}$ and $\text{ad}X(Y) = [X, Y]$. It is equivalent to the condition the metric g to be an invariant metric, i. e.

$$g([X, Y], Z) + g([X, Z], Y) = 0. \quad (15)$$

Theorem 2.1. *If (G, J, g) is an almost Norden manifold with a Killing metric g , then it is:*

- (i) a \mathcal{W}_3 -manifold;
- (ii) a locally symmetric manifold.

Proof. (i) Let ∇ be the Levi-Civita connection of g . Then the condition (15) implies consecutively

$$\nabla_{X_i} X_j = \frac{1}{2} [X_i, X_j], \quad i, j \in \{1, 2, \dots, 2n\}, \quad (16)$$

$$F(X_i, X_j, X_k) = \frac{1}{2} \left\{ g([X_i, JX_j], X_k) - g([X_i, X_j], JX_k) \right\}. \quad (17)$$

According to (15) the last equation implies $\mathfrak{S}_{X_i, X_j, X_k} F(X_i, X_j, X_k) = 0$, i. e. the manifold belongs to the class \mathcal{W}_3 .

(ii) The following form of the curvature tensor is given in Ref. 4

$$R(X_i, X_j, X_k, X_l) = -\frac{1}{4} g([X_i, X_j], [X_k, X_l]).$$

Using the condition (15) for a Killing metric, we obtain

$$R(X_i, X_j, X_k, X_l) = -\frac{1}{4} g([X_i, X_j], [X_k, X_l]). \quad (18)$$

According to the constancy of the component R_{ijkl} and (16) and (18), we get the covariant derivative of the tensor R of type $(0, 4)$ as follows

$$\begin{aligned} (\nabla_{X_i} R)(X_j, X_k, X_l, X_m) &= \frac{1}{8} \left\{ g([X_i, X_j], [X_k, X_l]) - g([X_i, X_k], [X_j, X_l]) - g([X_i, X_l], [X_j, X_k]) \right. \\ &\quad \left. + g([X_i, X_l], [X_m, X_j]) - g([X_i, X_m], [X_l, X_j]) - g([X_i, X_j], [X_m, X_l]) \right\}. \end{aligned} \quad (19)$$

We apply the the Jacobi identity (12) to the double commutators. Then the equation (19) gets the form

$$\begin{aligned} (\nabla_{X_i} R)(X_j, X_k, X_l, X_m) &= -\frac{1}{8} \left\{ g([X_i, [X_j, X_k]], [X_l, X_m]) \right. \\ &\quad \left. + g([X_i, [X_l, X_m]], [X_j, X_k]) \right\}. \end{aligned} \quad (20)$$

Since g is a Killing metric, then applying (15) to (20) we obtain the identity $\nabla R = 0$, i. e. the manifold is locally symmetric. \square

3. The Lie Group as a 6-Dimensional \mathcal{W}_3 -Manifold

Let (G, J, g) be a 6-dimensional almost Norden manifold with Killing metric g . Having in mind Theorem 2.1 we assert that (G, J, g) is a \mathcal{W}_3 -manifold. Let the commutators have the following decomposition

$$[X_i, X_j] = \gamma_{ij}^k X_k, \quad \gamma_{ij}^k \in \mathbb{R}, \quad i, j, k \in \{1, 2, \dots, 6\}. \quad (21)$$

Further we consider the special case when the following two conditions are satisfied

$$g([X_i, X_j], [X_k, X_l]) = 0, \quad g([X_i, JX_j], [X_i, JX_i]) = 0 \quad (22)$$

for all different indices i, j, k, l in $\{1, 2, \dots, 6\}$. In other words, the commutators of the different basis vectors are mutually orthogonal and moreover the commutators of the holomorphic sectional bases are isotropic vectors with respect to the Norden metric g .

According to the condition (15) for a Killing metric g , the Jacobi identity (12) and the condition (22), the equations (21) take the form given in Table 1.

Table 1. The Lie brackets with 3 parameters.

	X_1	X_2	X_3	X_4	X_5	X_6
$[X_2, X_3]$					λ_1	λ_2
$[X_3, X_1]$				λ_1		λ_3
$[X_1, X_2]$				λ_2	λ_3	
$[X_5, X_6]$		$-\lambda_1$	$-\lambda_2$			
$[X_6, X_4]$	$-\lambda_1$		$-\lambda_3$			
$[X_4, X_5]$	$-\lambda_2$	$-\lambda_3$				
$[X_1, X_4]$		λ_2	$-\lambda_1$		λ_2	$-\lambda_1$
$[X_1, X_5]$		λ_3		$-\lambda_2$		
$[X_1, X_6]$			$-\lambda_3$	λ_1		
$[X_2, X_4]$	$-\lambda_2$				λ_3	
$[X_2, X_5]$	$-\lambda_3$		λ_1	$-\lambda_3$		λ_1
$[X_2, X_6]$			λ_2		$-\lambda_1$	
$[X_3, X_4]$	λ_1					$-\lambda_3$
$[X_3, X_5]$		$-\lambda_1$				λ_2
$[X_3, X_6]$	λ_3	$-\lambda_2$		λ_3	$-\lambda_2$	

The Lie groups G thus obtained are of a family which is characterized by three real parameters λ_i ($i = 1, 2, 3$). Therefore, for the manifold (G, J, g) constructed above, we establish the truthfulness of the following

Theorem 3.1. *Let (G, J, g) be a 6-dimensional almost Norden manifold, where G is a connected Lie group with corresponding Lie algebra \mathfrak{g} determined by the global basis of left invariant vector fields $\{X_1, X_2, \dots, X_6\}$; J is an almost complex structure defined by (13) and g is an invariant Norden metric determined by (14) and (15). Then (G, J, g) is a quasi-Kähler manifold with Norden metric if and only if G belongs to the 3-parametric family of Lie groups determined by Table 1.*

Let us remark, the Killing form $B(X, Y) = \text{tr}(\text{ad}X \text{ad}Y)$, $X, Y \in \mathfrak{g}$, on

the constructed Lie algebra \mathfrak{g} has the following form

$$B = 4 \left(\begin{array}{c|c} L & -L \\ \hline -L & L \end{array} \right), \quad L = \begin{pmatrix} \lambda_3^2 & -\lambda_2\lambda_3 & -\lambda_1\lambda_3 \\ -\lambda_2\lambda_3 & \lambda_2^2 & -\lambda_1\lambda_2 \\ -\lambda_1\lambda_3 & -\lambda_1\lambda_2 & \lambda_1^2 \end{pmatrix}.$$

Obviously, it is degenerate.

4. Geometric characteristics of the constructed manifold

Let (G, J, g) be the 6-dimensional \mathcal{W}_3 -manifold introduced in the previous section.

4.1. The components of the tensor F

Then by direct calculations, having in mind (2), (13), (14), (15), (16), (17) and Table 1, we obtain the nonzero components of the tensor F as follows

$$\begin{aligned} \lambda_1 &= 2F_{116} = 2F_{161} = -2F_{134} = -2F_{143} = 2F_{223} = 2F_{232} \\ &= 2F_{256} = 2F_{265} = -F_{322} = -F_{355} = 2F_{413} = 2F_{431} \\ &= 2F_{446} = 2F_{464} = -2F_{526} = -2F_{562} = 2F_{535} = 2F_{553} \\ &= -F_{611} = -F_{644} = -2F_{113} = -2F_{131} = -2F_{146} = -2F_{164} \\ &= -2F_{226} = -2F_{262} = 2F_{235} = 2F_{253} = F_{311} = F_{344} = 2F_{416} \\ &= 2F_{461} = -2F_{434} = -2F_{443} = -2F_{523} = -2F_{532} = -2F_{556} \\ &= -2F_{565} = F_{622} = F_{655}, \end{aligned} \tag{23}$$

$$\begin{aligned} \lambda_2 &= -2F_{115} = -2F_{151} = 2F_{124} = 2F_{142} = F_{233} = F_{266} = -2F_{323} \\ &= -2F_{332} = -2F_{356} = -2F_{365} = -2F_{412} = -2F_{421} = -2F_{445} \\ &= -2F_{454} = F_{511} = F_{544} = -2F_{626} = -2F_{662} = 2F_{635} = 2F_{653} \\ &= 2F_{112} = 2F_{121} = 2F_{145} = 2F_{154} = -F_{211} = -F_{244} = -2F_{326} \\ &= -2F_{362} = 2F_{335} = 2F_{353} = -2F_{415} = -2F_{451} = 2F_{424} \\ &= 2F_{442} = -F_{533} = -F_{566} = 2F_{623} = 2F_{632} = 2F_{656} = 2F_{665}, \end{aligned} \tag{24}$$

$$\begin{aligned}
\lambda_3 &= -F_{133} = -F_{166} = -2F_{215} = -2F_{251} = 2F_{224} = 2F_{242} = 2F_{313} \\
&= 2F_{331} = 2F_{346} = 2F_{364} = -F_{422} = -F_{455} = 2F_{512} = 2F_{521} \\
&= 2F_{545} = 2F_{554} = 2F_{616} = 2F_{661} = -2F_{634} = -2F_{643} = F_{122} \\
&= F_{155} = -2F_{212} = -2F_{221} = -2F_{245} = -2F_{254} = 2F_{316} \\
&= 2F_{361} = -2F_{334} = -2F_{343} = F_{433} = F_{466} = -2F_{515} = -2F_{551} \\
&= 2F_{524} = 2F_{542} = -2F_{613} = -2F_{631} = -2F_{646} = -2F_{664}.
\end{aligned} \tag{25}$$

where $F_{ijk} = F(X_i, X_j, X_k)$.

4.2. The square norm of ∇J

According to (14) and (23)–(25), from (11) we obtain that the square norm of ∇J is zero, i. e. $\|\nabla J\| = 0$. Then we have the following

Proposition 4.1. *The manifold (G, J, g) is isotropic Kählerian.*

4.3. The components of R

Let R be the curvature tensor of type (0,4) determined by (7) and (6) on (G, J, g) . We denote its components by $R_{ijks} = R(X_i, X_j, X_k, X_s)$; $i, j, k, s \in \{1, 2, \dots, 6\}$. Using (16), (12), (18) and Table 1 we get the nonzero components of R as follows

$$\begin{aligned}
-R_{1221} &= R_{4554} = \frac{1}{4}(\lambda_2^2 + \lambda_3^2), \quad -R_{1551} = R_{2442} = \frac{1}{4}(\lambda_2^2 - \lambda_3^2), \\
-R_{1331} &= R_{4664} = \frac{1}{4}(\lambda_1^2 + \lambda_3^2), \quad -R_{1661} = R_{3443} = \frac{1}{4}(\lambda_1^2 - \lambda_3^2), \\
-R_{2332} &= R_{5665} = \frac{1}{4}(\lambda_1^2 + \lambda_2^2), \quad -R_{2662} = R_{3553} = \frac{1}{4}(\lambda_1^2 - \lambda_2^2), \\
R_{1361} &= R_{2362} = -R_{4364} = -R_{5365} = \frac{1}{4}\lambda_1^2, \\
R_{1251} &= R_{3253} = -R_{4254} = -R_{6256} = \frac{1}{4}\lambda_2^2, \\
R_{2142} &= R_{3143} = -R_{5145} = -R_{6146} = \frac{1}{4}\lambda_3^2,
\end{aligned}$$

$$\begin{aligned}
R_{1561} &= R_{2562} = R_{3563} = -R_{4564} = -R_{1261} = -R_{3263} \\
&= R_{4264} = R_{5265} = -R_{1351} = -R_{2352} = R_{4354} \\
&= R_{6356} = R_{1231} = -R_{4234} = -R_{5235} = -R_{6236} = \frac{1}{4}\lambda_1\lambda_2,
\end{aligned}$$

$$\begin{aligned}
-R_{1341} &= -R_{2342} = R_{5345} = R_{6346} = R_{2132} = -R_{4134} \\
&= -R_{5135} = -R_{6136} = R_{1461} = R_{2462} = R_{3463} \\
&= -R_{5465} = -R_{2162} = -R_{3163} = R_{4164} = R_{5165} = \frac{1}{4}\lambda_1\lambda_3,
\end{aligned}$$

$$\begin{aligned}
R_{3123} &= -R_{4124} = -R_{5125} = -R_{6126} = -R_{1241} = -R_{3243} \\
&= R_{5245} = R_{6246} = -R_{2152} = -R_{3153} = R_{4154} \\
&= R_{6156} = R_{1451} = R_{2452} = R_{3453} = -R_{6456} = \frac{1}{4}\lambda_2\lambda_3.
\end{aligned}$$

4.4. The components of ρ and the value of τ

Having in mind (8) and the components of R , we obtain the components $\rho_{ij} = \rho(X_i, X_j)$ ($i, j = 1, 2, \dots, 6$) of the Ricci tensor ρ and the value of the scalar curvature τ as follows

$$\begin{aligned}
\rho_{11} &= \rho_{44} = -\rho_{14} = -\lambda_3^2, & \rho_{12} &= -\rho_{15} = -\rho_{24} = \rho_{45} = \lambda_2\lambda_3, \\
\rho_{22} &= \rho_{55} = -\rho_{25} = -\lambda_2^2, & \rho_{13} &= -\rho_{16} = -\rho_{34} = \rho_{46} = \lambda_1\lambda_3, \\
\rho_{33} &= \rho_{66} = -\rho_{36} = -\lambda_1^2, & \rho_{23} &= -\rho_{26} = -\rho_{35} = \rho_{56} = \lambda_1\lambda_2,
\end{aligned} \tag{26}$$

$$\tau = 0. \tag{27}$$

The last equation implies immediately

Proposition 4.2. *The manifold (G, J, g) is scalar flat.*

4.5. The sectional curvatures

Let us consider the characteristic 2-planes α_{ij} spanned by the basis vectors $\{X_i, X_j\}$ at an arbitrary point of the manifold:

- holomorphic 2-planes - $\alpha_{14}, \alpha_{25}, \alpha_{36}$;
- pairs of totally real 2-planes - $(\alpha_{12}, \alpha_{45}); (\alpha_{13}, \alpha_{46}); (\alpha_{15}, \alpha_{24}); (\alpha_{16}, \alpha_{34}); (\alpha_{23}, \alpha_{56}); (\alpha_{26}, \alpha_{35})$.

Then, using (9), (14) and the components of R , we obtain the corresponding sectional curvatures

$$\begin{aligned}
k(\alpha_{14}) &= k(\alpha_{25}) = k(\alpha_{36}) = 0; \\
-k(\alpha_{12}) &= k(\alpha_{45}) = \frac{1}{4}(\lambda_2^2 + \lambda_3^2), & k(\alpha_{15}) &= -k(\alpha_{24}) = \frac{1}{4}(\lambda_2^2 - \lambda_3^2), \\
-k(\alpha_{13}) &= k(\alpha_{46}) = \frac{1}{4}(\lambda_1^2 + \lambda_3^2), & k(\alpha_{16}) &= -k(\alpha_{34}) = \frac{1}{4}(\lambda_1^2 - \lambda_3^2), \\
-k(\alpha_{23}) &= k(\alpha_{56}) = \frac{1}{4}(\lambda_1^2 + \lambda_2^2), & k(\alpha_{26}) &= -k(\alpha_{35}) = \frac{1}{4}(\lambda_1^2 - \lambda_2^2).
\end{aligned}$$

Therefore we have the following

Proposition 4.3. *The manifold (G, J, g) has zero holomorphic sectional curvatures.*

4.6. *The isotropic-Kählerian property*

Having in mind Proposition 4.1–Proposition 4.3 and Theorem 2.1, we give the following characteristics of the constructed manifold

Theorem 4.1. *The manifold (G, J, g) constructed as an \mathcal{W}_3 -manifold with Killing metric in Theorem 3.1:*

- (i) *is isotropic Kählerian;*
- (ii) *is scalar flat;*
- (iii) *is locally symmetric;*
- (iv) *has zero holomorphic sectional curvatures.*

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